2006

The Negs and Regs of Continued Fractions

Alexander Eustis
Harvey Mudd College

The Negs and Regs of Continued Fractions

Alex Eustis

Arthur T. Benjamin, Advisor

Sanjai Gupta, Reader

May, 2006

Harvey Mudd College
Department of Mathematics

Abstract

There are two main aims of this thesis. The first is to further develop and demonstrate applications of the combinatorial interpretation of continued fractions introduced in [Benjamin and Quinn, 2003]. The second is to investigate the theory of *negative* continued fractions, a relatively unresearched topic. That is, discuss the ways in which they are similar to and different from the regular class, describe how to convert between the two forms, and show that the central theorems concerning regular continued fractions also apply to the negative ones.

Acknowledgments

My advisor, Professor Arthur T. Benjamin, made this topic possible in the first place, and provided excellent suggestions and guidance. I'd also like to thank Sanjai Gupta, my second reader; Professor Darryl Yong for helpful LaTeXtips; and Professor Richard Guy for suggesting some research questions.

Contents

Chapter 1

Introduction to Continued Fractions

Continued fractions have been well-studied and have a rich history. They first appeared in the 16th century, although Brezinski [1991] argues that related concepts can be traced back to antiquity. Here we reproduce the basic definitions and theorems.

1.1 Notation and Basic Theory

Definition 1. *A* finite continued fraction *is an expression of the form*

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots + \frac{b_n}{a_n}}}}. \tag{1.1}$$

Usually a_i and b_i are integers, although they can be real numbers, complex numbers, polynomials, etc.

Definition 2. *A* regular continued fraction *is one in which*

- $b_k = 1$ *for all* k
- a_0 *is an integer*
- $a_1, a_2, \ldots$ *are positive integers.*

For regular continued fractions we use the notation

$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}. \tag{1.2}$$

The coefficients a_k are known as the *partial quotients* of the regular continued fraction.

Perhaps the most basic fact about continued fractions is that they serve as rational approximations to real numbers, in the same way that decimal expansions do. For decimal expansions, this is summed up by the following propositions:

- If $d_1, d_2, \ldots$ are integers in the range $0 \le d_i \le 9$, then the series

  $$\sum_{i=1}^{\infty} \frac{d_i}{10^i}$$

  converges to some $x \in [0, 1]$.

- For all $x \in [0, 1]$, there exist $d_1, d_2, \ldots$, $0 \le d_i \le 9$, such that

  $$\sum_{i=1}^{\infty} \frac{d_i}{10^i} = x.$$

  The representation is unique for irrational x, and for rational x there are at most two such representations.

For regular continued fractions, it is well known [Rockett and Szüsz, 1992] that analogous theorems hold.

Theorem 3. *Let $a_0, a_1, \ldots$ satisfy the criteria for a regular continued fraction. Then the sequence*

$$\{c_n\} = [a_0, a_1, \ldots, a_n]$$

always converges to some $x \in \mathbb{R}$. This limit is known as the infinite continued fraction $[a_0, a_1, \ldots]$ *and is always an irrational number. The rational numbers c_n are known as the* convergents *or* approximants.

Theorem 4. *If x is an irrational number, then there exist unique $a_0, a_1, \ldots$, satisfying the criteria for a regular continued fraction, such that*

$$\lim_{n \to \infty} [a_0, a_1, \ldots, a_n] = x.$$

If x is rational, then x has exactly 2 finite representations:

$$x = [a_0, \ldots, a_n]$$

for some a_i, with $a_n \geq 2$ (or $n = 0$); and

$$x = [a_0, \ldots, a_n - 1, 1].$$

The main idea for proving existence in Theorem 4 is that, given a real number x, the partial quotients for x may be calculated using the following "truncate down and reciprocate" algorithm.

1. Let $x_0 = x$, $k = 0$.
2. Let $a_k = \lfloor x_k \rfloor$.
3. If x_k is an integer, then terminate.
4. Otherwise, compute the truncated and reciprocated number
$$x_{k+1} = \frac{1}{x_k - a_k}$$
5. Set $k = k + 1$ and go back to step 2.

The invariant that makes this algorithm work is that each time we compute x_k in step 4 (or in step 1), we have $x = [a_0, a_1, \ldots, a_{k-1}, x_k]$. To see this, note that it is initially true in step 1 (since $x = [x_0] = [x]$). Now, assume that $x = [a_0, \ldots, a_{k-1}, x_k]$ for some k. We now compute a_{k+1} and x_{k+1} in steps 2 and 4 (respectively), and solving for x_k we have $x_k = a_k + \frac{1}{x_{k+1}}$. Therefore,

$$x = [a_0, \ldots, a_{k-1}, x_k] = [a_0, \ldots, a_{k-1}, a_k + \frac{1}{x_{k+1}}] = [a_0, \ldots, a_{k-1}, a_k, x_{k+1}]$$

as desired.

If x is rational, then the algorithm will eventually terminate at step 3. Otherwise it will never terminate, but we can use it to compute as many partial quotients a_k as we desire.

Example: Using the algorithm, we can calculate the first few partial quotients for $\pi = 3.14159\ldots$:

$$\begin{array}{lll}
a_0 = 3 & x_1 = \frac{1}{.14159\ldots} = 7.0625\ldots & \pi = 3 + \frac{1}{x_1} \\
a_1 = 7 & x_2 = \frac{1}{.0625\ldots} = 15.9966\ldots & \pi = 3 + \frac{1}{7 + \frac{1}{x_2}} \\
a_2 = 15 & x_3 = \frac{1}{.9966\ldots} = 1.0034\ldots & \pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{x_3}}}
\end{array}$$

Continuing in this manner, we have $\pi = [3, 7, 15, 1, \ldots]$. Therefore, the first convergents for π are $c_0 = 3$, $c_1 = [3, 7] = \frac{22}{7}$, $c_2 = [3, 7, 15] = \frac{333}{106}$, $c_3 = [3, 7, 15, 1] = \frac{355}{113}$. These rational numbers give very close approximations to π.

Although the next two theorems are typically used for the purpose of proving the previous two, they are of interest to us in their own right. This first theorem shows how to compute $[a_0, \ldots, a_n]$ "forwards," rather than backwards as order of operations would suggest.

Theorem 5. *Define A_k, B_k recursively as follows:*

$$\begin{array}{llll} A_{-1} & = 1, & B_{-1} & = 0 \\ A_0 & = a_0, & B_0 & = 1 \end{array}$$

$$\left.\begin{array}{ll} A_k & = a_k A_{k-1} + A_{k-2} \\ B_k & = a_k B_{k-1} + B_{k-2} \end{array}\right. \quad (k \geq 1).$$

Then

$$[a_0, a_1, \ldots, a_k] = \frac{A_k}{B_k}$$

in lowest terms.

Theorem 6. *If A_k and B_k are defined as above, then*

$$A_k B_{k-1} - A_{k-1} B_k = (-1)^{k-1}.$$

Equivalently,

$$\frac{A_k}{B_k} - \frac{A_{k-1}}{B_{k-1}} = \frac{(-1)^{k-1}}{B_k B_{k-1}}. \tag{1.3}$$

Theorem 6 shows that, since B_k increases at least as fast as the Fibonacci numbers, $\{\frac{A_k}{B_k}\}$ converges at least exponentially. It also shows that the convergents alternately approximate the limit from above and below.

In the next chapter, we will prove more general versions of these two theorems using a combinatorial approach.

1.2 Negative Continued fractions

In this paper we investigate an alternative definition, which is similar to the regular continued fraction in many ways.

Definition 7. *A* negative *continued fraction is one in which*

- *$b_k = -1$ for all k*
- *a_0 is an integer*
- *$a_1, a_2, \ldots$ are integers ≥ 2.*

For negative continued fractions we use the notation

$$[a_0, a_1, \ldots, a_n]_- = a_0 - \cfrac{1}{a_1 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}. \tag{1.4}$$

Negative continued fractions arise naturally if we change the "truncate down and reciprocate" algorithm from the previous section to "truncate *up* and reciprocate." That is, if we change steps 2 and 4 to use the ceiling instead of the floor as follows:

1. Let $x_0 = x$, $k = 0$.
2. Let $a_k = \lceil x_k \rceil$.
3. If x_k is an integer, then terminate.
4. Otherwise, compute the truncated and reciprocated number
$$x_{k+1} = \frac{1}{a_k - x_k}$$
5. Set $k = k + 1$ and go back to step 2.

This algorithm computes the partial quotients a_k in such a way that $x = [a_0, a_1, \ldots]_-$, which means that Theorem 4 has a direct analogue for negative continued fractions. Also, Theorem 3 (convergence) is no less elementary for negative continued fractions than for regular ones. We shall see that the other two theorems from the previous section carry over to negative continued fractions, as well.

Chapter 2

Combinatorial Interpretation of Continued Fractions

2.1 Square-and-domino tilings

We now define the main combinatorial object in this paper, which is a *square-and-domino tiling*. In this chapter, we shall see that the weighted versions of these tilings are directly related to continued fractions. Once we have developed some facts about tilings and proved their connection to continued fractions, we will be in a position to analyze continued fractions from a combinatorial standpoint.

This approach is based on chapter 4 of [Benjamin and Quinn, 2003], but in this paper we significantly extend these ideas. In what follows, we focus on the new results, giving a rapid treatment of the old results where necessary.

Definition 8. *Let n be a nonnegative integer. Consider a board of length n, or "n-board" for short. At our disposal we have square tiles of length 1, and domino tiles of length 2. A* square-and-domino tiling *is an arrangement of tiles that covers the board completely with no overlapping tiles. Let $\mathcal{F}_n$ denote the set of all square-and-domino tilings of the n-board.*

Example: When $n = 4$ there are five square-and-domino tilings, as shown in Figure 2.1.

Theorem 9. *Let $F_0 = 0, F_1 = 1, F_2 = 1, F_2 = 2, F_3 = 3, \ldots$ be the Fibonacci sequence. Then for all $n \geq -1$,*

$$|\mathcal{F}_n| = F_{n+1}.$$

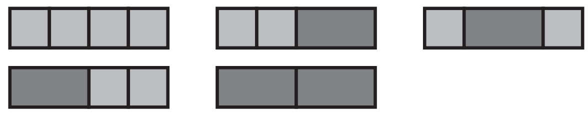


Figure 2.1: All five tilings of the 4-board.

Figure 2.2: Layout of weights for a weighted tiling.

This theorem explains the notation $\mathcal{F}_n$. It is proved by showing that $|\mathcal{F}_n| = |\mathcal{F}_{n-1}| + |\mathcal{F}_{n-2}|$, which follows by considering whether the last tile is a square or a domino.

2.1.1 Weighted tilings

Next, we introduce the idea of a weighted tiling and some associated notation. Consider an $(n+1)$-board with cells labeled 0 to n. For $1 \leq k \leq n$, we also define the *kth boundary* to be the boundary between cells $k-1$ and k. Since a domino always covers one of these boundaries, we use the phrase "domino on boundary k" as our standard terminology rather than the more cumbersome "domino covering cells $k-1$ and k."

To give *weights* to the board means that we assign real numbers $a_0, \ldots, a_n$ to the cells and real numbers $b_1, \ldots, b_n$ to the boundaries, as shown in Figure 2.2.

When a weighted board is tiled with squares and dominoes, each tile is assigned a weight according to its location. That is, a square on cell k gets weight a_k and a domino on boundary k gets weight b_k. Using these tile weights, we make the following definition:

Definition 10. *Let $a_0, b_1, a_1, b_2, \ldots, b_n, a_n$ be the weights of an $(n+1)$-board. Given a tiling $T \in \mathcal{F}_{n+1}$, define its weight $w(T)$ to be the product of the weights of its tiles. We also define the* weighted sum *of the tilings in $\mathcal{F}_{n+1}$ and notate it as follows:*

$$|0:n| = \sum_{T \in \mathcal{F}_{n+1}} w(T) \tag{2.1}$$

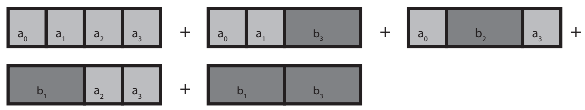


Figure 2.3: All the ways to tile the $|0:3|$ board.

Whenever we use the notation $|0:n|$, the weights $a_0, b_1, \ldots, b_n, a_n$ must be implied from context when they are not shown explicitly.
Example 1: A simple example is when $a_i = b_i = 1$ for all i. Then each tiling has unit weight, and so $|0:n| = F_{n+2}$ by Theorem 9.
Example 2: When $n = 3$, there are five possible tilings, whose weights are shown in Figure 2.3. It follows that

$$|0:3| = a_0a_1a_2a_3 + a_0a_1b_3 + a_0b_2a_3 + b_1a_2a_3 + b_1b_3.$$

For integers $0 \leq i \leq j \leq n$, we also write $|i:j|$ to mean the weighted sum of the tilings of the sub-board starting at cell i and ending at cell j. We'll also denote the sub-board itself as $i:j$. For example, $|i:i| = a_i$ and $|i:i+1| = a_ia_{i+1} + b_{i+1}$. It is also consistent and convenient to define $|i:i-1| = 1$ and $|i:i-2| = 0$, since we may think of the first quantity as the empty product (a tiling with no tiles) and the second as the empty sum (no allowed tilings at all). The identities we prove will still hold using these conventions.

2.1.2 Weighted Tiling Identities

The two identities in this section will be our most important tools for working with weighted tilings. The first identity shows how $|0:n|$ can be computed from $|0:-1| = 1$, $|0:-2| = 0$, and the following recurrence.

Identity 11. *For $n \geq 0$,*

$$|0:n| = a_n|0:n-1| + b_n|0:n-2|.$$

Proof. There are two types of tilings: those that end in a square and those that end in a domino. Consider the sum of the weights of all tilings ending in a square. These weights all have a common factor of a_n which distributes out of the sum, leaving us with the weights of all the tilings of the $0:n-1$ sub-board. This gives the $a_n|0:n-1|$ term. The same argument shows that the weighted sum of all tilings ending in a domino is $b_n|0:n-2|$. □

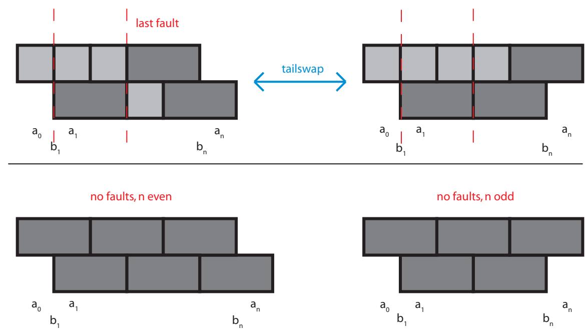


Figure 2.4: Possible and impossible tailswaps

When $a_i = b_i = 1$, this becomes $F_{n+2} = F_{n+1} + F_n$, the familiar recurrence relation for Fibonacci numbers.

There are other forms of this identity. If we consider the first tile rather than the last, we obtain

$$|0 : n| = a_0|1 : n| + b_1|2 : n|. \tag{2.2}$$

Also, we can generalize the identity by applying it in the middle of the board, or noting that it applies equally well to sub-boards. The most general version is when we consider whether a domino lies on the kth boundary of an $i : j$ sub-board (for $i \leq k \leq j$):

$$|i : j| = |i : k-1||k : j| + b_k|i : k-2||k+1 : j| \tag{2.3}$$

In fact, it is true in general that any theorem about $|0 : n|$ is also a theorem about $|i : j|$ because there is nothing special about starting the indices at 0.

Next is the "tailswapping" identity, named for the method of its proof.

Identity 12. *For $n \geq 1$,*

$$|0 : n-1||1 : n| = |0 : n||1 : n-1| + (-1)^n \prod_{i=1}^{n} b_i.$$

Proof. It follows from Definition 10 that $|0 : n-1||1 : n|$ is the weighted sum of all *pairs* of tilings in the staggered formation shown on the left side of Figure 2.4. $|0 : n||1 : n-1|$ is the same thing for the slightly different formation on the right side. As before, the weight of a tiling pair is the product of the weights of all the individual tiles. Let A denote the set of all tiling pairs for sub-boards $0 : n-1$ and $1 : n$, and let B denote the set of all tiling pairs for $0 : n$ and $1 : n-1$.

To prove the theorem, we will show that A and B can *almost* be put into a bijective, weight-preserving correspondence. The meaning of "almost" is that there is precisely one tiling pair left out of the correspondence, as we shall see.

Recall that for $1 \le k \le n$, we defined the kth boundary to be the boundary between cells $k-1$ and k. A tiling pair is said to have a *fault line* on the kth boundary if neither tiling in the pair has a domino covering that boundary. Visually, this means that the tiling pair has a vertical line running through it, as shown in Figure 2.4.

If a given tiling pair has a fault line, then we define its *tail* to be the part that lies past the last fault line. By swapping the top and bottom portions of the tail, we obtain a map from A to B or B to A as shown in the diagram. This map is its own inverse because it preserves the location of the last fault line. It also preserves weight because all the tiles always stay in their original column.

There is only one fault-free tiling pair: the one consisting of all dominoes. Its weight is $\prod b_i$, and it belongs to A if n is even and B if n is odd. This accounts for the $(-1)^n \prod b_i$ term, completing the proof.

□

2.1.3 The weighted sum as a determinant

We can also view this weighted sum as the determinant of a certain matrix. Several authors (for instance, Clarke et al. [1999]) use this determinant instead of a continued fraction. The theorem below shows that the two approaches are equivalent.

Theorem 13. *Let $a_0, b_1, a_1, b_2, \ldots, b_n, a_n$ be any real numbers. Then*

$$|0:n| = \begin{vmatrix} a_0 & -1 & 0 & \cdots & 0 \\ b_1 & a_1 & -1 & & \vdots \\ 0 & b_2 & a_2 & & 0 \\ \vdots & & & \ddots & -1 \\ 0 & \cdots & 0 & b_n & a_n \end{vmatrix}.$$

Since this theorem is not crucial to this paper, its proof may be skipped on a first reading.

Proof. Denote this matrix by P, and for $-1 \le k \le n$ let P_k denote the matrix formed by removing all the rows and columns of P numbered greater than k. For instance, $P_0 = [a_0]$ and $P_n = P$. P_{-1} is the empty matrix, whose determinant is 1 by convention. We wish to show that $|0 : k| = P_k$ for all $-1 \le k \le n$. In particular, $|0 : n| = |P_n| = |P|$.

To do this, we prove that $|0 : k|$ and P_k satisfy the same initial conditions and recurrence relation. The initial conditions are trivial: $|P_{-1}| = |0 : -1| = 1$ and $|P_0| = |0 : 0| = a_0$.

For the recurrence relation we shall use DeMorgan's rule for expanding a determinant recursively. Examine the following block diagram for the matrix P_k:

$$P_k = \left[\begin{array}{c|cc} P_{k-2} & & \\ & -1 & \\ \hline b_{k-1} & a_{k-1} & -1 \\ & b_k & a_k \end{array}\right]$$

Consider taking the determinant of P_k by expanding across the last column. When we do this there are two nonzero terms. Clearly one of the terms is $a_k|P_{k-1}|$. To compute the other term we would start with the coefficient -1, add a minus sign as according to the determinant rule, and multiply by the determinant of the matrix with column k and row $k-1$ removed. *This* matrix has only the single entry b_k in its bottom row, so its determinant is $b_k|P_{k-2}|$. Altogether we have $|P_k| = a_k|P_{k-1}| + b_k|P_{k-2}|$, which is the same recurrence as Identity 11. □

2.2 Continued fractions

2.2.1 General Case

The following theorem gives the connection between continued fractions and combinatorics. More specifically, it shows how to express a continued fraction in terms of the weighted sums defined above. This theorem will allow us to study continued fractions by looking at their corresponding tilings.

Theorem 14. *Let $a_0, b_1, a_1, b_2, \ldots, b_n, a_n$ be any real numbers. Then*

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{\ddots + \cfrac{b_n}{a_n}}} = \frac{|0 : n|}{|1 : n|}.$$

Furthermore, if $a_i \in \mathbb{Z}$ and $b_i = \pm 1$, then $\frac{|0:n|}{|1:n|}$ is in lowest terms.

Proof. We prove the stronger-looking theorem that for any $0 \leq i \leq j \leq n$,

$$a_i + \cfrac{b_{i+1}}{a_{i+1} + \cfrac{b_{i+2}}{\ddots + \frac{b_j}{a_j}}} = \frac{|i:j|}{|i+1:j|}. \tag{2.4}$$

The proof is by induction on the length of the continued fraction, i.e. $j - i$. In the base case ($j = i$), we have the trivial equation $a_i = \frac{a_i}{1} = \frac{|i:i|}{|i+1:i|}$. Now, assume that the formula holds for length less than $j - i$; then we may write

$$a_i + \cfrac{b_{i+1}}{a_{i+1} + \cfrac{b_{i+2}}{\ddots + \frac{b_j}{a_j}}} = a_i + \frac{b_{i+1}}{|i+1:j|/|i+2:j|} = \frac{a_i|i+1:j| + b_{i+1}|i+2:j|}{|i+1:j|}$$

$$= \frac{|i:j|}{|i+1:j|}, \tag{2.5}$$

where the last equality follows by letting $k = i + 1$ in Equation (2.3) (the general version of Identity 11).

For the second part of the theorem, when a_i are integers and $b_i = \pm 1$, we need to prove that $|0:n|$ and $|1:n|$ are relatively prime integers. We can prove this by induction on n. The base case $n = 0$ is clear; $\gcd(a_0, 1) = 1$. For the inductive step, we can use Equation (3.8) and the Euclidean Algorithm:

$$\gcd(|0:n|, |1:n|) = \gcd(a_0|1:n| \pm |2:n|, |1:n|) = \gcd(|2:n|, |1:n|),$$

which equals one by our inductive hypothesis. □

Example:

We will use Fibonacci numbers as an example once more. By the previous theorem, the sequence

$$1, \quad 1 + \frac{1}{1}, \quad 1 + \cfrac{1}{1 + \frac{1}{1}}, \quad 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}, \quad \ldots$$

may be written in lowest terms as F_2/F_1, F_3/F_2, $F_4/F_3, \ldots$. This is one way to see that the infinite continued fraction $[1, 1, \ldots]$ is equal to the golden ratio ϕ.

At this point, it is worth revisiting Theorems 5 and 6 from Chapter 1, whose proofs were postponed. In the statements of those theorems, it is readily seen (from checking the initial conditions and recurrence relation) that $A_k = |0:k|$ and $B_k = |1:k|$ when $b_i = 1$ for all i. Therefore, Theorem 5 is a special case of Theorem 14, and similarly, Theorem 6 is the same special case of Identity 12.

If x is real number whose regular continued fraction has partial quotients $a_0, a_1, \ldots$, then by Theorem 14, $\frac{|0:k|}{|1:k|}$ is the kth convergent to x. Therefore, the equation

$$\frac{|0:k|}{|1:k|} - \frac{|0:k-1|}{|1:k-1|} = \frac{(-1)^{k-1}}{|1:k||1:k-1|}$$

(which comes from Identity 12) proves that the convergents alternately approximate x from above and below, as we mentioned in Chapter 1. For negative continued fractions, we get

$$\frac{|0:k|}{|1:k|} - \frac{|0:k-1|}{|1:k-1|} = \frac{1}{|1:k||1:k-1|},$$

which proves that the convergents of a *negative* continued fraction always approximate x from above. We'll have more to say about this in the next chapter.

2.2.2 Integer values

So far, we have only assumed that $a_0, \ldots, a_n, b_1, \ldots, b_n$ are real numbers. In fact, there is nothing to stop us from considering them to be complex numbers or in fact elements of any commutative ring. However, for our purposes they will usually be integers. When a_k, b_k are nonnegative integers, we can give an interpretation for $|0:n|$ which is even more combinatorial; it will be the size of a set rather than just a sum of weights. The objects in the set are still square-and-domino tilings, but we do away with the weights and instead allow squares and dominoes to be stacked.

Definition 15. *Let $a_0, \ldots, a_n$ and $b_1, \ldots, b_n$ be nonnegative integers. We define a* tiling of an $(n+1)$-board with height conditions *as follows. Every cell from 0 to n is covered by a "stack of squares" or a "stack of dominoes." Specifically, cell k can be covered by a stack of squares up to height a_k. On boundary k, we allow dominoes to be stacked up to height b_k.*

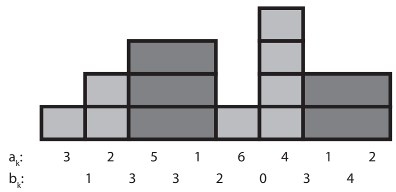


Figure 2.5: Stacked tilings are valid as long as the stack heights do not exceed the number a_k or b_k in that location. The stack of squares on cells 1 and 5 are at their maximum height, as is the stack of dominoes on boundary 3.

Of course, we don't allow squares to be stacked on top of dominoes or any other such combinations. Figure 2.5 shows an example of a tiling that satisfies the given height restrictions.

Theorem 16. *For positive integers $a_0, \ldots, a_n$ and nonnegative integers $b_1, \ldots, b_n$, the number of tilings of an $n+1$-board with these height conditions is equal to $|0:n|$.*

Proof. For any tiling $T \in \mathcal{F}_{n+1}$, the product of the tiles $w(T)$ gives the number of different ways to form a height-restricted tiling whose square and domino stacks occur in the same places as the squares and dominoes (respectively) in T. Thus, by Equation (10), $\sum_{T\in\mathcal{F}_{n+1}} w(T) = |0:n|$ gives the total number of height-restricted tilings overall. □

2.2.3 Negative Dominoes

In order to deal with negative continued fractions, we also want to know how to generalize Theorem 16 to interpret $|0:n|$ as the size of a set even when $b_i < 0$.

Before we define the tilings that will make up this set, we first give an intuitive interpretation. For a moment consider the trivial case $b_k = 0$ for all k. With no dominoes allowed, it is clear that $|0:n| = \prod_{k=0}^{n} a_k$. Now, for any given $1 \le k \le n$, letting b_k be positive gives us b_k *more* ways to tile the pair of cells $k-1, k$. If we want b_k to be negative, then instead of allowing dominoes, we should impose a restriction on these all-square tilings so that there will be $|b_k|$ *fewer* ways to tile cells $k-1, k$. For instance, we could disallow the following combination: a maximum height stack in cell $k-1$ followed by any stack of height $\le |b_k|$ in cell k. We establish this restriction formally in the next definition.

Definition 17. *Let $a_0, \ldots, a_n$ be positive integers, and let $b_1, \ldots, b_n$ be integers satisfying $a_k > -b_k$ for $1 \leq k \leq n$. Then we define a* mixed tiling of an $(n+1)$-board with height conditions $a_0, b_1, a_1, \ldots, b_n, a_n$ *as follows. In cell k we allow stacks of squares up to height a_k. If $b_k \geq 0$ then on boundary k we also allow stacks of dominoes up to height b_k, as usual. If $b_k < 0$ then dominoes are not allowed on boundary k. Additionally, we impose the following* directionality requirement*.*

1. *If $b_k < 0$ and cell $k-1$ contains a stack of squares of maximum height (that is, of height a_{k-1}), then cell $k-1$ is called* right-facing*. We denote a right-facing cell by $\rightarrow$.*
2. *If $b_k < 0$ and cell k contains a stack of squares of height 1 up to $-b_k$, then this stack is called* left-facing*. We denote a left-facing cell by $\leftarrow$.*
3. *The directionality requirement prohibits cells from facing each other; that is, $\rightarrow\leftarrow$ is forbidden.*

Let us note a few properties of the directionality requirement. If $b_k < 0$, then on cell $k-1$ there is one way to have $\rightarrow$, and on cell k there are $|b_k|$ ways to have $\leftarrow$. (This uses the fact that $a_k \geq -b_k$). Therefore, the directionality requirement prohibits a total of $|b_k|$ combinations that give $\rightarrow\leftarrow$.

Also note that if $b_k < 0$ and $b_{k+1} < 0$, then cell k is eligible for both left-facing and right-facing status. However, the requirement $a_k > -b_k$ prohibits it from satisfying both at once. Why? If cell k is right-facing, then it contains a stack of a_k squares. If it is also left-facing, then it contains a stack of $\leq -b_k$ squares. This implies that $a_k \leq -b_k$, which contradicts our requirement. So, a stack of squares can be left-facing, right-facing, or neither, but not both.

We now prove that Theorem 16 generalizes fully to mixed tilings.

Theorem 18. *Let $a_0, \ldots, a_n, b_1, \ldots, b_n$ are integers satisfying $a_k \geq 1$, $a_k > -b_k$ ($1 \leq k \leq n$). Then the number of mixed tilings of an $n+1$-board with these height conditions is $|0 : n|$.*

Proof. We can show that the number of mixed tilings satisfies the same initial conditions and recurrence relation as $|0 : n|$. Let us denote this number by $M(0 : n)$. We establish the usual conventions of $M(0 : -2) = 0 = |0 : -2|$ and $M(0 : -1) = 1 = |0 : -1|$. Now we must prove that for all $n \geq 0$,

$$M(0 : n) = a_n M(0 : n-1) + b_n M(0 : n-2).$$

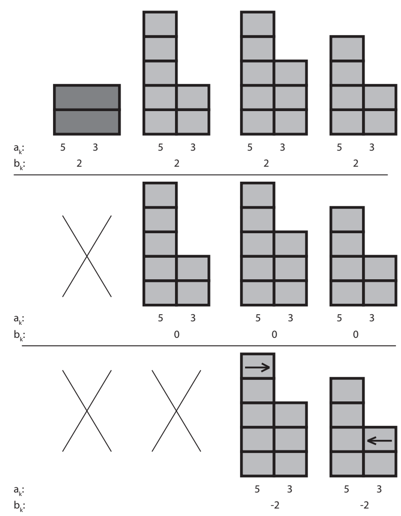


Figure 2.6: As b_k decreases, the tilings allowed near boundary k become more restricted.

If $b_n \geq 0$, then this recurrence is seen by considering whether cell n contains squares or dominoes, just as before.

The new situation is when $b_n < 0$. In this case, the last cell always contains a square stack up to height a_n. Therefore there are $a_n M(0 : n-1)$ ways to tile the board, disregarding the directionality requirement on the last two cells. Next, we must subtract off the tilings that fail the directionality requirement; i.e. those tilings with $\rightarrow\leftarrow$ in the last two cells. As discussed earlier, there are $|b_k| = -b_k$ ways to have $\rightarrow\leftarrow$ in the last two cells, and $M(0, n-2)$ ways to tile the rest of the board, for a total of $-b_n M(0, n-2)$ such tilings. Subtracting these tilings from the over-counted total, we obtain $M(0, n) = a_n M(0, n-1) + b_n M(0, n-2)$ as desired. $\square$

2.2.4 A word on $a_k > -b_k$

This section is somewhat technical and may be omitted on a first reading. In the above proof, recall that we needed the restriction $a_k > -b_k$ so that when $b_k < 0$, the following facts would be true:

1. $|b_k|$ is the number of ways to have $\rightarrow\leftarrow$ in cells $k-1, k$
2. Cell k cannot simultaneously count as both $\rightarrow$ and $\leftarrow$.

Now, item 1 above only requires that $a_k \geq -b_k$. Furthermore, item 2 is only relevant if cell k is actually eligible for $\rightarrow$ status, which means that $k < n$ and $b_{k+1} < 0$. In the case that $k = n$ or $b_{k+1} \geq 0$, the requirement can actually be relaxed to $a_k \geq -b_k$ without affecting the truth of any propositions in this paper. To summarize, in place of the restriction $a_k > -b_k$ (for $1 \leq k \leq n$) we can use the following, slightly more lenient condition:

$$a_k > -b_k - I(b_{k+1} \geq 0),$$

where $I(b_{k+1} \geq 0)$ is the indicator function defined as

$$I(b_{k+1} \geq 0) = \begin{cases} 1, & k = n \text{ or } b_{k+1} \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Although this restriction may seem arbitrarily imposed in order to make certain theorems in this paper true, this condition actually arises naturally when we consider continued fractions that are built using a generalized "round and reciprocate" algorithm. We investigate this now.

Suppose we are given some $x \in \mathbb{R}$ that we want to approximate by a continued fraction, but this time we are also given a sequence of nonzero integers $b_1, b_2, \ldots$, which we want to be the numerators of the continued fraction. For instance, if $b_k = 1$ for all k, then we are asking for the regular continued fraction for x, and if $b_k = -1$ for all k, then we are asking for the negative continued fraction. In this more general situation, we can use the following algorithm to come up with the partial quotients a_k:

1. Let $x_0 = x$, $k = 0$.
2. If $b_{k+1} > 0$, let $a_k = \lfloor x_k \rfloor$. Otherwise, if $b_{k+1} < 0$, let $a_k = \lceil x_k \rceil$.
3. If x_k is an integer, then terminate.
4. Let
$$x_{k+1} = \frac{b_{k+1}}{x_k - a_k}$$
5. Set $k = k + 1$ and go back to step 2.

As discussed in Chapter 1, the invariant driving this algorithm is that each time we compute the newest x_k, we have

$$x = [a_0, a_1, \ldots, a_{k-1}, x_k],$$

where the continued fraction in the above equation is interpreted to have numerators given by the $\{b_i\}$ sequence.

In the two special cases of $b_k = 1$ for all k and $b_k = -1$ for all k, the reader can easily verify that the above algorithm coincides with the ones presented in Chapter 1. We now prove that the partial quotients output by this algorithm will always satisfy the restriction we just defined, making it therefore "natural" in some sense.

Proposition 19. *Fix some $x \in \mathbb{R}$, and a sequence of nonzero integers $b_1, b_2, \ldots$. If a_k is a partial quotient calculated by the above algorithm, and $k \geq 1$, then*

$$a_k > |b_k| - I(b_{k+1} \geq 0),$$

where

$$I(b_{k+1} \geq 0) = \begin{cases} 1, & b_{k+1} \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. To prove the inequality for some given $\hat{k} \geq 1$, consider the steps starting from Step 2 when $k = \hat{k} - 1$.

- Step 2 calculates the unique integer $a_{\hat{k}-1}$ such that $x_{\hat{k}-1} - a_{\hat{k}-1}$ has magnitude less than 1, and the same sign as $b_{\hat{k}}$.
- Step 3 does not terminate the algorithm (since we've yet to calculate $a_{\hat{k}}$).
- Therefore, $x_{\hat{k}}$ as calculated in Step 4 will be strictly greater than $|b_{\hat{k}}|$.
- Step 2 of the next iteration will calculate $a_{\hat{k}}$. No matter what, we'll have that $a_{\hat{k}} \geq \lfloor x_{\hat{k}} \rfloor \geq |b_{\hat{k}}|$.
- In the case when $I(b_{\hat{k}+1} \geq 0) = 0$, we must also prove that $a_{\hat{k}} \geq |b_{\hat{k}}| + 1$. This follows from the facts that $a_{\hat{k}} = \lceil x_{\hat{k}} \rceil$ and $x_{\hat{k}} > b_{\hat{k}}$.

□

Since $|b_k| \geq -b_k$, this proves that the coefficients a_k output by the algorithm will meet the restrictions for a mixed tiling. This includes the last coefficient (if there is one), which is only required to satisfy $a_n \geq -b_n$.

2.2.5 Mixed tilings with $b_k = \pm 1$

Although the tilings in the previous section allowed $|b_k| > 1$, often we are only concerned with $b_k = \pm 1$. For $b_k = \pm 1$, we adopt another notation for $|0 : n|$ in which the values a_k and b_k are expressed explicitly. When $b_k = 1$ we use the symbol $\oplus$, and when $b_k = -1$ we use the symbol $\ominus$. For example, the notation

$$|a_0 \oplus a_1 \ominus a_2| \tag{2.6}$$

means the number of ways to tile a mixed board with height conditions $a_0, b_1 = 1, a_1, b_2 = -1, a_2$.

We prefer a slightly different interpretation for this type of tiling. Recall that cell k can have a_k possible heights of square stacks. If the left boundary of cell k (i.e. b_k) is -1, then one of these stacks gets the label $\leftarrow$, and if the right boundary is -1, then one of the stacks gets the label $\rightarrow$. Instead of thinking of $\leftarrow$ and $\rightarrow$ as labels that can be applied to square stacks, we could consider them to be *objects unto themselves* with which we may tile the board. Under this interpretation, the maximum height of a square stack in cell k is no longer a_k; it is a_k *minus* the number of negative boundaries of cell k, which may be 0, 1, or 2.

For example, consider the board $|5 \ominus 3 \oplus 2|$. Cell 0 allows square stacks up to height 4, or $\rightarrow$. Cell 1 allows square stacks up to height 2, or $\leftarrow$, or the left half of a domino, which we denote $\sqsubset$. Cell 2 allows square stacks up to height 2, or $\sqsupset$. The rule governing these objects is that $\rightarrow\leftarrow$ must never appear together, and $\sqsubset\sqsupset$ must always appear together.

Notice that in the situation when $a_k = 2$ and $b_k = b_{k+1} = -1$, cell k only allows $\leftarrow$ or $\rightarrow$; never any squares or dominoes at all.

Chapter 3

Continued Fractions and Combinatorics

In the previous chapter, we defined $|0 : n|$ as a weighted sum of square-and-domino tilings, and we proved that continued fractions and weighted sums are related by the formula

$$[0 : n] = \frac{|0 : n|}{|1 : n|}.$$

When $a_0, \ldots, a_n, b_1, \ldots, b_n$ are integers subject to the restrictions $a_k \geq 1$, $a_k > -b_k$, we also gave a combinatorial interpretation for $|0 : n|$ as the number of *mixed tilings with height conditions*. We are now ready to prove facts about continued fractions using combinatorics. This chapter will demonstrate how this may be done.

In particular, we focus on negative and mixed continued fractions, since the approach in Benjamin and Quinn's Proofs That Really Count only applies to regular continued fractions. This chapter only provides a sample of the problems we could attack. The Future Work section indicates some other topics that could be addressed.

3.1 How to Convert between Regs and Negs

If we are going to study negative continued fractions, a natural question to ask is how they are related to regular continued fractions. For $x \in \mathbb{R}$, there exists, in fact, a direct relationship between the partial quotients for the regular continued fraction of x and those of the negative continued fraction of x. In what follows we shall describe and prove this relationship.

We begin with a theorem that will allow us to convert a small section of a board that uses dominoes ($\oplus$) into a section that uses arrows ($\ominus$) without changing the number of tilings.

Theorem 20. *Let $a_0, a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be integers satisfying the usual conditions $a_k \geq 0$ and $b_k > -a_k$, and in addition, $b_k = \pm 1$. Then*

$$\begin{aligned} |\ldots a_{m-1} \oplus a_m \oplus a_{m+1} \ldots| &= \\ |\ldots (a_{m-1}+1) \ominus \underbrace{2 \ominus 2 \ominus 2}_{a_m - 1 \text{ times}} \ominus (a_{m+1}+1) \ldots|. \end{aligned} \tag{3.1}$$

In this equation, the ellipses on the left are placeholders for the unexpressed values $a_0, b_1, a_1, \ldots, b_{m-1}, a_{m-2}$. These values must be the same for both sides of the identity. Similar comments apply to the ellipses on the right.

Proof. We want to find a one-to-one correspondence between tilings of the first board and of the second. To do so we need a bit of terminology for board 2. The sequences of 2s is called the *arrow region* because the only objects allowed there are $\leftarrow$ and $\rightarrow$. The cell immediately to the left of the arrow region (with height condition $a_{m-1}+1$) is the *left border* and the cell to the right is the *right border*. We now provide a dictionary which we use to map the cells $m-1$, m, and $m+1$ of board 1 onto the left border, arrow region, and right border (respectively) of board 2.

The only object allowed in cell $m-1$ of board 1 that is not allowed in the left border of board 2 is the left half of a domino, which we abbreviate $\sqsubset$. Also, the only object allowed in the left border of board 2 that is not allowed in cell $m-1$ of board 1 is $\rightarrow$. This naturally suggests that we use the rule

$$\sqsubset \quad \longmapsto \quad \rightarrow \tag{3.2}$$

for this cell. Any other object appearing in cell $m-1$ of board 1 may be replaced by the same object in board 2, including square stacks of equal heights.

Similarly, we use the rule

$$\sqsupset \quad \longmapsto \quad \leftarrow \tag{3.3}$$

for cell $m+1$. It should be noted that rules 3.2 and 3.3 cannot both be invoked within the same tiling, because this would require two dominoes to overlap in cell m.

For cell m, rules 3.3 and 3.2 force us to use the following in order to satisfy directionality:

$$\sqsubset \quad \longmapsto \quad \underbrace{\leftarrow \ldots \leftarrow}_{a_m-1} \tag{3.4}$$

$$\sqsupset \quad \longmapsto \quad \underbrace{\rightarrow \ldots \rightarrow}_{a_m-1} \tag{3.5}$$

The remaining possibility is that a stack of squares appears in cell m. We observe that the number of ways to tile the arrow region is exactly a_m because we must begin with some number of $\leftarrow$ between 0 and $a_m - 1$ inclusive, and the remaining cells (if any) must be $\rightarrow$. This suggests that for a stack of h squares, $\boxed{\text{h}}$, we should use the rule

$$\boxed{\text{h}} \quad \longmapsto \quad \underbrace{\leftarrow \ldots \leftarrow}_{h-1}\underbrace{\rightarrow \ldots \rightarrow}_{a_m-h}. \tag{3.6}$$

This never breaks directionality because we already covered the cases in which $\rightarrow$ or $\leftarrow$ appear in the border squares.

The reader may be concerned about the degenerate case $a_m = 1$, where the arrow region is empty. However, this is treated no differently. Since rules 3.4, 3.5, and 3.6 tell us to put 0 arrows in the arrow region, we will do exactly that! Moreover, having $\rightarrow\leftarrow$ in the left/right borders is still impossible by the overlapping dominoes argument above.

This completes the description of how to map cells $m-1$, m, and $m+1$. Of course the rest of the board is mapped to itself without any changes. Now that we have defined the correspondence between board 1 and board 2, it is really no more work to see that the correspondence is in fact a bijection. This is because if we start with a tiling of board 2, we can simply apply the above rules in reverse to determine a unique tiling of board 1. The only ambiguity is when the arrow region contains all arrows of the same kind; this might be mapped onto either half a domino or a stack of squares of height 1 or a_m. But, whether or not one *more* arrow appears in the appropriate border square distinguishes which case to apply. For example, when $a_{m-1} = 5$, $a_m = 3$, and $a_{m+1} = 2$, $\boxed{4}\leftarrow\leftarrow\leftarrow$ (board 2) must have come from $\boxed{4}\sqsubset\sqsupset$ (board 1) , whereas $\boxed{4}\leftarrow\leftarrow\boxed{1}$ must have come from $\boxed{4}\boxed{3}\boxed{1}$. $\square$

3.1.1 A conversion formula

What we investigate next is that when Theorem 20 is used repeatedly, we can convert an entire board with $b_k = 1$ to one with $b_k = -1$. First, we note a simple modification that can be made to the end of a board.

Identity 21.

$$|a_0 \ldots| = |1 \oplus (a_0 - 1) \ldots|, \tag{3.7}$$

where the ellipses again refer to the unspecified values: $b_1, a_1, b_2, \ldots, a_n$.

Proof. The correspondence between boards 1 and 2 is very simple. We'll say that board 2 begins on cell -1. If cell 0 of board 1 is a maximal stack of squares, then replace it with a domino covering cells -1 and 0; otherwise leave it be and add a square in cell -1. □

Applied to the other end of the tiling, this identity states

$$|\ldots a_n| = |\ldots \oplus (a_n - 1) \oplus 1|. \tag{3.8}$$

Theorem 20 and the above identity give us a procedure for converting between regular and negative continued fractions.

For example, consider the regular continued fraction

$$6 + \cfrac{1}{2 + \frac{1}{3 + \frac{1}{4}}}.$$

By Theorem 14, this may be rewritten as

$$\frac{|6 \oplus 2 \oplus 3 \oplus 4|}{|2 \oplus 3 \oplus 4|}.$$

We can now use Theorem 20 and Equation (3.8) to convert these two reg-boards into similar neg-boards:

$$\begin{array}{rll} & |6 \oplus 2 \oplus 3 \oplus 4| & \\ = & |6 \oplus 2 \oplus 3 \oplus 3 \oplus 1| & \text{(Equation 3.8)} \\ = & |7 \ominus 2 \ominus 4 \oplus 3 \oplus 1| & \text{(Theorem 20)} \\ = & |7 \ominus 2 \ominus 5 \ominus 2 \ominus 2 \ominus 2| & \text{(Theorem 20)}, \end{array}$$

and

$$\begin{array}{rll} & |2 \oplus 3 \oplus 4| & \\ = & |1 \oplus 1 \oplus 3 \oplus 3 \oplus 1| & \text{(Equation 3.8)} \\ = & |2 \ominus 4 \oplus 3 \oplus 1| & \text{(Theorem 20)} \\ = & |2 \ominus 5 \ominus 2 \ominus 2 \ominus 2| & \text{(Theorem 20)} \end{array}$$

Thus we conclude that

$$6 + \cfrac{1}{2 + \frac{1}{3 + \frac{1}{4}}} = \frac{|6 \oplus 2 \oplus 3 \oplus 4|}{|2 \oplus 3 \oplus 4|} = \frac{|7 \ominus 2 \ominus 5 \ominus 2 \ominus 2 \ominus 2|}{|2 \ominus 5 \ominus 2 \ominus 2 \ominus 2|} = 7 - \cfrac{1}{2 - \cfrac{1}{5 - \cfrac{1}{2 - \cfrac{1}{2 - \frac{1}{2}}}}}.$$

This same procedure can be made to work for any regular continued fraction. We prove this in the next few theorems. But first, some notation that should help us.

Definition 22. *If $a_0, \ldots, a_n$ is a finite sequence of positive integers, then we let $N(a_0, \ldots, a_n)$ be another finite sequence, called the* neg-sequence for $a_0, \ldots, a_n$, *defined as*

$$N(a_0, \ldots, a_n) = (a_0 + 1), \underbrace{2, \ldots, 2}_{a_1 - 1}, (a_2 + 2), \underbrace{2, \ldots, 2}_{a_3 - 1}, (a_4 + 2), \ldots.$$

This sequence ends in either $(a_n + 1)$ or $\underbrace{2, \ldots, 2}_{a_n - 1}$, according to the parity of n. If $a_0, a_1, \ldots$ is an infinite sequence, we define $N(a_0, a_1, \ldots)$ according to the same alternating pattern.

Theorem 23. *Let $a_0, \ldots, a_n$ be a sequence of positive integers. Let the terms of the neg-sequence $N(a_0, \ldots, a_n)$ be denoted as $\bar{a}_0, \ldots, \bar{a}_l$, and let the terms of the so-called "alternate neg-sequence" $a_0, N(a_1, a_2, \ldots, a_n)$ be denoted as $a_0, \bar{a}'_1, \bar{a}'_2, \ldots, \bar{a}'_m$. Then*

(a)

$$|a_0 \oplus \ldots \oplus a_n| = |\bar{a}_0 \ominus \bar{a}_1 \ominus \bar{a}_2 \ominus \ldots \ominus \bar{a}_l| = |a_0 \oplus \bar{a}'_1 \ominus \bar{a}'_2 \ominus \ldots \ominus \bar{a}'_m|.$$

(b)

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_n}}} = \bar{a}_0 - \cfrac{1}{\bar{a}_1 - \cfrac{1}{\ddots - \frac{1}{\bar{a}_l}}} = \bar{a}'_0 + \cfrac{1}{\bar{a}'_1 - \cfrac{1}{\ddots - \frac{1}{\bar{a}'_m}}}.$$

Proof of (a). The idea for this proof is demonstrated in the above examples. Beginning with the board $|a_0 \oplus \ldots \oplus a_n|$, make the following changes to it:

1. If necessary, apply Equation (3.8) to the right edge so that there are an even number of $\oplus$ signs altogether (odd for the alternate neg sequence).

2. Starting with the first two $\oplus$ signs (second and third for the alternate), apply Theorem 20 to this and each subsequent pair to convert them to $\ominus$.

□

Proof of (b). By Theorem 14, we need to show that

$$|a_0 \oplus a_1 \oplus \ldots \oplus a_n| = |\bar{a}_0 \ominus \bar{a}_1 \ominus \ldots \ominus \bar{a}_l| = |\bar{a}'_0 \oplus \bar{a}'_1 \ominus \ldots \ominus \bar{a}'_m|$$

and

$$|a_1 \oplus \ldots \oplus a_n| = |\bar{a}_1 \ominus \ldots \ominus \bar{a}_l| = |\bar{a}'_1 \ominus \ldots \ominus \bar{a}'_m|.$$

The first equation is Theorem 23 verbatim. The second equation states that

$$|a_1 \oplus a_2 \oplus \ldots \oplus a_n| = |\underbrace{2 \ominus \ldots \ominus 2}_{a_1 - 1} \ominus (a_2 + 2) \ominus \ldots| = |(a_1 + 1) \ominus \underbrace{2 \ominus \ldots \ominus 2}_{a_2 - 1} \ominus \ldots|,$$

which is merely a slight variation on Theorem 23. The sequence appearing on the right is the neg-sequence for $a_1, \ldots, a_n$ and the sequence appearing in the middle is the neg-sequence for $1, (a_1 - 1), a_2, a_3, \ldots, a_n$, which is equivalent to $a_1, \ldots, a_n$ by Identity 21. □

We are now ready to prove the main result of this section, which shows that the neg-sequence is in fact the sequence of partial quotients for the negative continued fraction.

Corrolary 24. *Let $x \in \mathbb{R}^+$, and let $a_0, a_1, \ldots$ be the (finite or infinite) sequence of partial quotients for the regular continued fraction of x. Then the sequence of partial quotients for the* negative *continued fraction of x is $N(a_0, \bar{a}_1, \ldots)$, and the sequence of partial quotients for the negative continued fraction of $-x$ is $-a_0, N(a_1, a_2, \ldots)$.*

Proof. This follows by taking the limit as $n \to \infty$ in part (b) of the previous theorem, and noting that

$$-x = -\left(a_0 + \cfrac{1}{\bar{a}'_1 - \cfrac{1}{\ddots}}\right) = -a_0 - \cfrac{1}{\bar{a}'_1 - \cfrac{1}{\ddots}}$$

□

3.1.2 Convergents

The results we have obtained so far allow us to say quite a bit about the regular/negative convergents of a continued fraction. We begin with another corollary that follows from the development in the previous section.

Corrolary 25. *Let $x \in \mathbb{R}^+$, and suppose*

- $c_0, c_1, \ldots$ *are the convergents for the regular continued fraction of* x
- $d_0, d_1, \ldots$ *are the convergents for the negative continued fraction of* x
- $-d'_0, -d'_1, \ldots$ *are the convergents for the negative continued fraction of* $-x$.

Then, for all odd n there exists some m such that $c_n = d_m$. For even n, there exists m such that $c_n = d'_m$.

Proof. Let $a_0, a_1, \ldots$ be the sequence of (regular) partial quotients for x. As in Theorem 23 let $\bar{a}_0, \bar{a}_1, \ldots = N(a_0, a_1, \ldots)$, and $\bar{a}'_1, \bar{a}'_2, \ldots = N(a_1, a_2, \ldots)$. Then, By Corollary 24, we have

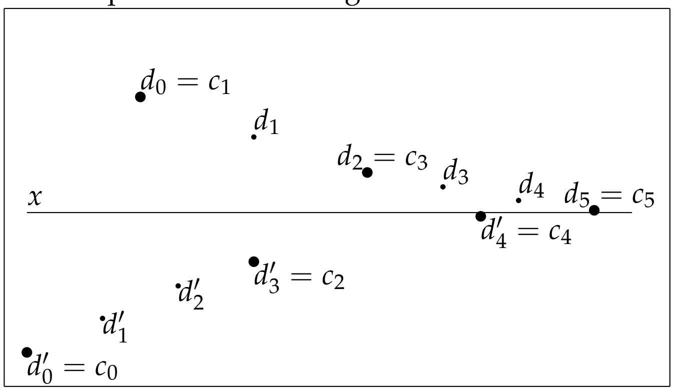
$$\begin{aligned} c_k &= [a_0, \ldots, a_k] \\ d_k &= [\bar{a}_0, \ldots, \bar{a}_k]_- \\ -d'_k &= [-a_0, \bar{a}'_1, \ldots, \bar{a}'_k]_- \end{aligned}$$

for all k.

Consider an index n. By Theorem 23, we have $c_n = [a_0, \ldots, a_n] = [N(a_0, \ldots, a_n)]_-$. Suppose that the finite neg-sequence $N(a_0, \ldots, a_n)$ has $m+1$ terms. Are these the same as $\bar{a}_0, \ldots, \bar{a}_m$, the first $m+1$ terms of $N(a_0, a_1 \ldots)$? A quick look at Definition 22 reveals that they are, *except possibly the last term*. If n is even, then $\bar{a}_m = a_n + 2$, whereas the last term of $N(a_0, \ldots, a_n)$ is $a_n + 1$. However, if n is odd, both sequences end in the same number of 2s, and are therefore identical. It follows that $c_n = d_m$ when n is odd.

A similar argument shows that $c_n = d'_m$ for n even. □

As we know from section 2.2.1, the convergents $c_0, c_1, c_2, \ldots$ alternately underestimate and overestimate x. On the other hand, $d_0, d_1, \ldots$ always overestimate x. Furthermore, $-d'_0, -d'_1, \ldots$ always overestimate $-x$, which means that $d'_0, d'_1, \ldots$ underestimate x. The new information given by the above corollary is that $d_0, d_1 \ldots$ include $c_1, c_3, \ldots$ as a subsequence, and $d'_0, d'_1, \ldots$ include $c_0, c_2, \ldots$ as a subsequence. This means that if we write the negative continued fractions for x and $-x$, we get *all* the convergents from the regular continued fraction, plus some additional convergents. The situation is pictured in the diagram below.



This naturally leads to the question: what are the extra convergents that we obtain from negative continued fractions? Our combinatorial interpretation can answer this question as well. First, an example.

Let $x = 2711/1731$. The first four convergents for the regular continued fraction of x are $c_0 = \frac{1}{1}, c_1 = \frac{2}{1}, c_2 = \frac{3}{2}, c_3 = \frac{11}{7}$. The first four convergents for the negative continued fraction of x are $d_0 = \frac{2}{1}, d_1 = \frac{5}{3}, d_2 = \frac{8}{5}, d_3 = \frac{11}{7}$. As expected, the neg-convergents include $\frac{2}{1}$ and $\frac{11}{7}$, but they also include two fractions in between. The sequence of numerators is $2, 5, 8, 11$, and the sequence of denominators is $1, 3, 5, 7$, which are both arithmetic progressions. In fact, they are the longest arithmetic progressions (of integers) that can go from 2 to 11 and from 1 to 7 respectively, such that both progressions have the same number of terms.

This suggests that, given the reg-convergents $\frac{2}{1}$ and $\frac{11}{7}$, we could have predicted the neg-convergents that appear in between. By calculating

$$\gcd(11 - 2, 7 - 1) = 3,$$

we realize that the longest such pair of arithmetic progressions has three steps, and so we would guess the correct sequence of convergents.

Indeed, we can prove that this works in general. First, we need to use a result from elementary number theory:

Lemma 26. *If $a, b, c, d \in \mathbb{Z}$ and $ab - cd = \pm 1$, then* $\gcd(a - c, b - d) = 1$.

Proof. Observe that

$$(a - c)(b + d) + (a + c)(b - d) = 2(ab - cd) = \pm 2,$$

so all we have to show is that $a - c$ and $b - d$ cannot both be even. If they were, then so would be $a + c$ and $b + d$, and so the left side of the above equation would be divisible by 4 whereas the right side (± 2) would not. This contradiction proves the claim. $\square$

Now, the theorem about the "in-between" convergents is as follows.

Theorem 27. *Let $c_{2n-1} = \frac{p_1}{q_1}, c_{2n+1} = \frac{p_2}{q_2}$ be two consecutive odd convergents of the regular continued fraction for x, and let $d_0, d_1, \ldots$ be the sequence of convergents of the negative continued fraction for x. From Corollary 25, $c_{2n-1} = d_m$ for some m, and $c_{2n+1} = d_{m+g}$ for some g.*

In addition,

$$g = gcd(p_2 - p_1, q_2 - q_1), \tag{3.9}$$

and the numerators and denominators of $d_m, d_{m+1}, \ldots, d_{m+g}$ each form an arithmetic progression.

The analogous result holds when we compare the even convergents c_{2n}, c_{2n+2} to $d'_0, d'_1, \ldots$ (the neg-convergents to $-x$).

Proof. Let $a_0, a_1, \ldots$ be the sequence of partial quotients for the regular continued fraction for x. For $0 \leq j \leq g$, write $d_{m+j} = e_j / f_j$ in lowest terms. From the previous theorems, we know how to express e_j and f_j in terms of the a_i as follows.

$$\begin{array}{llllllll} e_j = & |(a_0+1) & \ominus & 2\ldots2 & \ominus\ldots\ominus & 2\ldots2 & \ominus & (a_{2n}+2)\ominus 2\ldots2| \\ f_j = & & & |\underbrace{2\ldots2}_{a_1-1} & \ominus\ldots\ominus & \underbrace{2\ldots2}_{a_{2n-1}-1} & \ominus & \underbrace{(a_{2n}+2)\ominus 2\ldots2}_{j}|. \end{array}$$

The above notation means that, of the terms within the final brace, only the first j are present.

First, we show that $e_0, \ldots, e_g$ and $f_0, \ldots, f_g$ are arithmetic progressions. Using Identity 11 we have, for $2 \leq j \leq g$,

$$\begin{aligned} e_j &= 2e_{j-1} - e_{j-2} \\ \Rightarrow e_j - e_{j-1} &= e_{j-1} - e_{j-2}. \end{aligned} \tag{3.10}$$

Thus, the difference between consecutive terms remains constant. The same argument holds for f_j.

Now, we have to show that $g = \gcd(p_2 - p_1, q_2 - q_1)$. Recall that $p_1 = e_0$, $p_2 = e_g$, $q_1 = f_0$, $q_2 = f_g$. Also, by the general formula for an arithmetic progression,

$$e_g = e_0 + (e_1 - e_0)g \qquad f_g = f_0 + (f_1 - f_0)g, \tag{3.11}$$

so

$$\gcd(p_2 - p_1, q_2 - q_1) = \gcd(g(e_1 - e_0), g(f_1 - f_0)) = g \gcd(e_1 - e_0, f_1 - f_0). \tag{3.12}$$

However, e_0/f_0 and e_1/f_1 are consecutive convergents of a negative continued fraction. So, by Identity 12,

$$e_0 f_1 - e_1 f_0 = 1, \tag{3.13}$$

and therefore $\gcd(e_1 - e_0, f_1 - f_0) = 1$ by Lemma 26. This shows that $\gcd(p_2 - p_1, q_2 - q_1) = g$ as desired.

For even convergents, the proof is identical except for the fact that the beginnings of the above tilings look a little different. We leave the details to the reader. □

3.2 An uncounted identity and generalizations

3.2.1 Original identity

Another application of the combinatorial framework for regular and negative continued fractions is a new proof of the following identity:

Identity 28. *Let F_n denote the nth Fibonacci number and L_n denote the nth Lucas number. Then, for all $m, n \in \mathbb{Z}^+$,*

$$\frac{F_{(n+1)m}}{F_{nm}} = L_m - \cfrac{(-1)^m}{L_m - \cfrac{(-1)^m}{\ddots - \frac{(-1)^m}{L_m}}},$$

where the term L_m appears n times in the continued fraction.

This identity (along with many others) appeared in the "Uncounted identities" section of [Benjamin and Quinn, 2003], challenging the reader to discover a combinatorial proof where the authors had not yet done so. By using our new combinatorial interpretation for negative continued fractions, this problem becomes tractable.

First, we want to convert the above identity to a statement about integers. Using Theorem 14, the right-hand side can be written in lowest terms as $\frac{|0:n-1|}{|1:n-1|}$, where $a_i = L_m$ and $b_i = (-1)^{m+1}$ for all i. Although $\frac{F_{(n+1)m}}{F_{nm}}$ is *not* in lowest terms (in general), one can show that

$$\frac{F_{(n+1)m}/F_m}{F_{nm}/F_m}$$

is. This leads us to conjecture the following identity, which is equivalent to the one above, but can be proven by direct combinatorial means.

Identity 29. *For all $n, m \in \mathbb{Z}^+$,*

$$F_{nm} = |0 : n-2| F_m,$$

where the weights of $|0 : n-2|$ are given by $a_i = L_m$, for $0 \leq i \leq n-2$, and $b_i = (-1)^{m+1}$ for $1 \leq i \leq n-2$.

Before proving this identity, we note that it is interesting in its own right. A well-known fact about Fibonacci numbers is that F_m divides F_{nm}. The above identity not only proves this, but also shows that the quotient is $|0 : n-2|$. This quotient can be represented in other forms as well: for instance, Benjamin and Quinn [2003] proved that

$$F_{nm} = F_m \sum_{j=1}^{n} F_{m-1}^{j-1} F_{m(n-j)+1},$$

and Benjamin and Rouse [2004] proved that

$$F_{(n+1)m} = F_m \sum_{x_1=0}^{n} \sum_{x_2=0}^{n} \cdots \sum_{x_m=0}^{n} \binom{n-x_m}{x_1} \binom{n-x_1}{x_2} \cdots \binom{n-x_{m-1}}{x_m}.$$

In fact, we will be borrowing some of the terminology from this latter paper, namely "open" and "closed."

To prove Identity 29, we need to use the combinatorial interpretations for F_n and L_n. For completeness we briefly state these here, but for more details see [Benjamin and Quinn, 2003].

Theorem 30. *Let n be a positive integer. Then F_n is the number of ways to tile of an $(n-1)$-board with squares and dominoes. (This was Theorem 9.)*

Furthermore, L_n is the number of ways to tile a circular ring of length n, called a Lucas n-bracelet, *with squares and dominoes. A tiling of this ring in which a domino covers cells $0, n$ is called* out-of-phase, *and there are F_{n-1} such tilings. Otherwise, the tiling is* in-phase, *and there are F_{n+1} such tilings.*

Proof of Identity 29. Our strategy will be to describe two sets whose sizes are equal to each side of the identity, then find a bijection between the sets. First we consider the case where m is odd, and then we discuss how the even case differs.

Set 1: F_{nm} counts square-and-domino tilings of length $nm-1$. However, for reasons that will become apparent, we always add half a domino

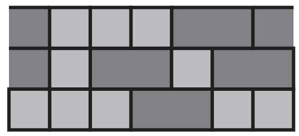


Figure 3.1: Example element of Set 1, for $n = 3$, $m = 7$.

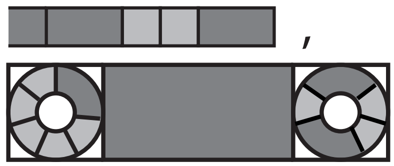


Figure 3.2: Example element of Set 2, for $n = 5$, $m = 7$. S is a 7-tiling beginning with a half-domino, and T is a 4-tiling, consisting of a square, a (rather large) domino, and another square, where each square is assigned a 7-bracelet.

to the beginning, increasing the length to nm, and we display the tiling in the format of n rows of length m (even if this causes some dominoes to appear split apart).

Set 2: $F_m|0 : n-2|$ Counts pairs of tilings (S, T), where S is an ordinary Fibonacci tiling of length $m-1$, to which we prepend a half-domino as above, and T is a height-conditioned tiling of length $n-1$ where each cell can have a stack of squares up to height L_m. By mapping the L_m possible square stack heights onto Lucas m-bracelets, we reinterpret T to consist of *Lucas m-bracelets* and dominoes, rather than square stacks and dominoes.

Correspondence: We begin with some additional terminology. The rows of Set 1 differ from ordinary tilings in that they can have half-dominoes on either end. Such ends are called *open*; otherwise they are *closed*. Also notice that, since the rows are derived from one continuous tiling, the following *continuity* properties are satisfied:

1. Row 1 is open on the left (due to our convention of beginning with a half-domino).

2. For $1 \leq i < n$, row i is open on the right if and only if row $i+1$ is open the left.

3. Row n is closed on the right

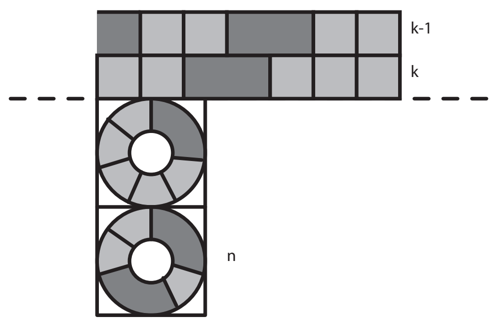


Figure 3.3: A tiling split by the dotted line.

We now describe a procedure to convert an element of Set 1 into an element of Set 2. We envision this procedure as a dotted line that begins just underneath the last row and moves upwards, eventually ending between rows 1 and 2. As the line moves, we will prove that it maintains the following invariant:

(a) The k rows above the line form a continuous F_{km} tiling. That is, they satisfy the three continuity properties above, with k in place of n.

(b) Below the line is a length $n - k$ tiling of Lucas m-bracelets and dominoes, as described under Set 2.

We already have noted that (a) is satisfied at the beginning of the procedure (when $k = n$), and (b) is satisfied vacuously because there is nothing below the line. At the end of the procedure, when $k = 1$, we will have an $(m-1)$-tiling above the line and a $0 : n-2$-tiling below it; that is, we will have an element of Set 2.

Now, we show how to move the dotted line upwards using a reversible transformation. If there is only one row above the line then we are already done. Otherwise, this transformation will involve tailswapping the two rows just above the line. Either this tailswap is possible or it is not.

Let us first consider the case when it is. Due to the continuity property, the end of row $k-1$ matches the beginning of row k. Therefore, after the tailswap is complete, row k has matched ends. This means that we can join these ends together to form a Lucas bracelet; it will be out-of-phase if the ends were open, or in-phase if they were closed. Then we move the dotted line upward one space as shown, to accommodate the Lucas bracelet we just created. This preserves all the continuity properties; in particular, 3

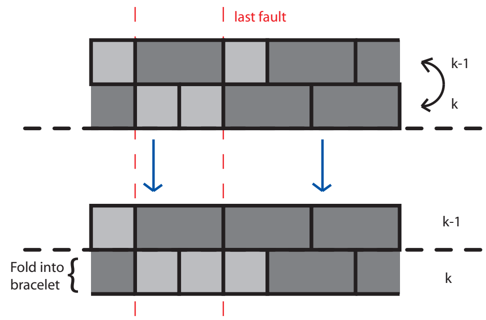


Figure 3.4: Tailswapping example.

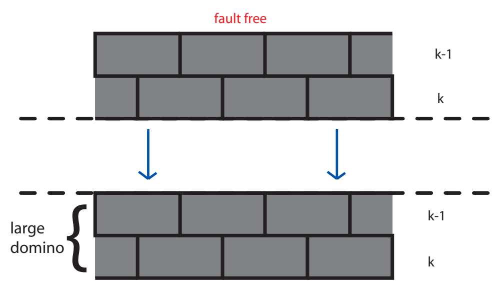


Figure 3.5: The fault-free configuration.

remains true because the closed end moved from row k to row $k-1$. (See Figure 3.4)

Now we consider the case when the pair of rows above the dotted line is fault-free, making the tailswap impossible. Since m is odd and row k is closed on the right, this can only occur from the staggered domino formation in Figure 3.5.

Notice that row $k-1$ is closed on the left, so it cannot be the first row. It follows that $k \geq 3$, and row $k-2$ is closed on the right. Therefore we may move the dotted line up *two* spaces, and we don't have to worry about jumping over the "finish line" $k = 1$ by doing so. As the reader may have guessed, the staggered formation left below the line is interpreted as a (large) domino in Set 2.

Because tailswapping is reversible, it follows that the entire procedure is reversible as well. We can depict any element of Set 2 as above, with the dotted line at $k = 1$. Then we move it downwards by applying the

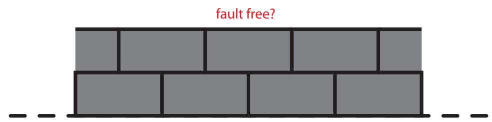


Figure 3.6: When m is even, the fault-free configuration is discontinuous.

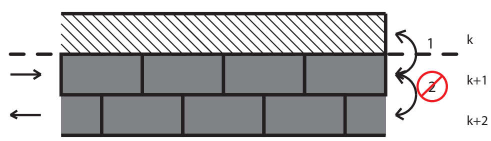


Figure 3.7: $\rightarrow\leftarrow$ cannot be the result of two consecutive tailswaps.

above steps in reverse. That is, if there is a Lucas bracelet just below the line, then we unhook it, tailswap, and move the line down one space. (The reader should verify that tailswapping is always possible in this direction.) If there is a domino, then replace it with the staggered formation and move the line down two spaces. This completes the proof of the identity for m odd.

When m is even, we must make a few changes to the proof, but it remains largely the same. The $|0 : n-2|$ term now has $b_k = -1$, so it represents a domino-free tiling consisting of Lucas bracelets, where we must choose a particular one of these bracelets to represent $\rightarrow$ and another to represent $\leftarrow$. As one might guess, the all-domino bracelets turn out to be the right choices; let the in-phase domino bracelet represent $\rightarrow$ and let the out-of-phase domino bracelet dominoes represent $\leftarrow$.

The procedure is largely the same as before; the only difference is the behavior of the exceptional fault-free case. When m is even, the fault-free arrangement would have to appear as in Figure 3.6.

However, this arrangement fails to satisfy continuity property 2, so it cannot be encountered. We conclude that tailswaps are *always* possible when m is even, so the procedure never uses large dominoes as required. We also have to check that it cannot leave $\rightarrow\leftarrow$ behind. We can prove this by contradiction; suppose our last two tailswaps resulted in $\rightarrow\leftarrow$. Then we would have the configuration in Figure 3.7

If we want, we should be able to undo these two tailswaps using the reverse procedure described earlier. When we attempt this on the above configuration, the last fault line of the first tailswap occurs at the very end

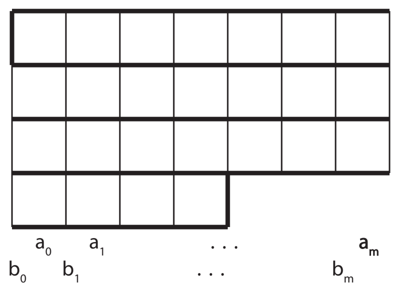


Figure 3.8: A periodic board with period $m + 1$. Any dominoes that are split between two rows are given the weight b_0.

of the rows, meaning that the first trivial exchange of empty tilings. This leaves us with an impossible second tailswap, contradicting what was just proved above; that tailswaps are always possible when m is even. □

3.2.2 Adding weights

As we have seen throughout this paper, Fibonacci identities are often special cases of weighted tiling identities. As it turns out, the Fibonacci and Lucas numbers in Identity 29 can be generalized by adding weights to the diagram. When we do this, Identity 29 generalizes to a theorem about *periodic* continued fractions.

Definition 31. *A sequence of weights $a_0, b_1, a_1, \ldots, b_n, a_n$ is* periodic with period *m if $a_j = a_{j+m}$ and $b_k = b_{k+m}$, for all $0 \le j \le n - m$ and $1 \le k \le n - m$.*

When defining the weights for an n-board with period m, it is enough to specify only the weights a_0 through a_{m-1} and b_1 through b_m. However, we usually refer to b_m using the name b_0 instead. As shown in Figure 3.8, we can visualize a periodic board using rows of length m, where tiles in the same column have the same weight.

Now, consider what happens when we consistently add weights to all the tilings used in the proof of Identity 29, according to the layout in Figure 3.8. In this tailswapping-based proof, tiles can only change position within their original column. Therefore, the Correspondence in the proof is weight-preserving, provided that we use a periodic board for Set 1.

When we add weights, it also becomes necessary to change the F_m and $|0 : n - 2|$ terms appearing in the proof. The F_m term just represents row 1, which begins with a half-domino; this becomes an ordinary weighted tiling $|1 : m-1|$ using the same weights from Set 1.

In Identity 29, the squares of the $|0 : n-2|$ board corresponded to Lucas bracelets. When we add weights, the in-phase bracelets will have total weight $|0 : m-1|$ and the out-of-phase bracelets will have total weight $b_0|1 : m-2|$. Therefore L_m, which was the weight of each square in the $|0 : n-2|$ board, generalizes to $|0 : m-1| + b_0|1 : m-2|$.

The dominoes of the $|0 : n-2|$ board corresponded to pairs of rows that were filled with dominoes in staggered formation. In the weighted scheme, the weight of such an object is $\prod_{i=0}^{m-1} b_i$. Therefore, the $(-1)^{m+1}$ term generalizes to $(-1)^{m+1}\prod_{i=0}^{m-1} b_i$.

Now, we are finally ready to state the weighted version of Identity 29.

Identity 32. *Let $a_0, \ldots, a_{m-1}$ and $b_0, \ldots, b_{m-1}$ be real numbers, and let $n, m \in \mathbb{Z}^+$. Then,*

$$|1 : nm-1| = |1 : m-1||0 : n-2|$$

where $|1 : nm-1|$ represents a periodic board of period m beginning with a half-domino, using the weights $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1}$; $|1 : m-1|$ is a tiling that uses the same weights; and $|0 : n-2|$ has the constant, "composite" weights of $a'_k = |0 : m-1| + b_0|1 : m-2|$, $b'_k = (-1)^{m+1}\prod_{i=0}^{n} b_i$ for all k.

Proof. As mentioned above, this is just the weighted version of Identity 29. We leave it to the reader to go through the proof again, verifying that when all tiles are assigned a weight according to their column, the argument is still valid. □

Let us also prove a variation on the previous identity:

Identity 33. *Let $a_0, \ldots, a_{m-1}$ and $b_0, \ldots, b_{m-1}$ be real numbers, and let $n, m \in \mathbb{Z}^+$. Then,*

$$|0 : nm-1| = |0 : n-1|$$

where $|0 : nm-1|$ represents a periodic board of period m, using the weights $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1}$; and $|0 : n-1|$ has the "composite" weights $a'_0 = |0 : m-1|$, $a'_k = |0 : m-1| + b_0|1 : m-2|$ for $1 \le k \le n-1$, and $b'_k = (-1)^{m+1}\prod_{i=0}^{n} b_i$ for all $1 \le k \le n-1$.

Proof. The proof is mostly the same as for Identity 32, with the following differences:

- The "finish line" is now at $k = 0$ instead of $k = 1$.

- The periodic tiling $|0 : nm - 1|$ no longer begins with a half-domino. Thus, continuity property 1 is changed to read: Row 1 in *closed* on the left.
- When m is odd, we may now encounter the fault-free configuration (that is, a domino) in rows 1 and 2. Hence it is possible to skip from $k = 2$ to $k = 0$.
- If we do reach $k = 1$, there is only one row above the dotted line, so we cannot tailswap. However, according to the new continuity property, the row $k = 1$ is closed on both ends, so it folds into an in-phase Lucas bracelet, and we finish the correspondence by moving the dotted line up to $k = 0$. It is not possible to create an out-of-phase Lucas bracelet in the first row.

□

3.2.3 Period-compression for periodic continued fractions

Now that we have expressions for the periodic weighted sums $|0 : nm - 1|$ and $|1 : nm - 1|$ (Identities 32 and 33), we can obtain a theorem about periodic continued fractions by rewriting the quotient $\frac{|0:nm-1|}{|1:nm-1|}$. For simplicity, we will only consider regular periodic continued fractions (i.e. $b_i = 1$ for all i), although negative continued fractions are no harder.

Theorem 34. *Let $a_0, a_1, \ldots, a_{m-1}$ be the weights for an infinite periodic continued fraction of period m, which is written as*

$$[\overline{a_0, a_1, \ldots, a_{m-1}}].$$

Then,

$$[\overline{a_0, a_1, \ldots, a_{m-1}}] = [a_0, a_1, \ldots, a_{m-1}] \pm \frac{1}{|1 : m-1|[L, L, \ldots]_{\pm}}$$

where $L = |0 : m-1| + |1 : m-2|$, and the subscript $\pm$ denotes a regular continued fraction when m is odd, and a negative continued fraction when m is even.

That is, we can write any periodic continued fraction in terms of its finite continued fraction based on one period, and a periodic continued fraction with period 1. In a sense, this theorem "compresses" the period to 1.

Proof. First consider the finite periodic continued fraction where the sequence $a_0, a_1, \ldots, a_{m-1}$ is repeated n times, which we denote $[(a_0, a_1, \ldots, a_{m-1})^n]$, and let m be odd. Then, according to identities 32 and 33, and Theorem 14,

$$[(a_0, a_1, \ldots, a_{m-1})^n] = \frac{|\,|0:m-1| \oplus L \oplus \ldots \oplus L\,|}{|1:m-1||L \oplus \ldots \oplus L|} = \frac{[|0:m-1|, L, \ldots, L]}{|1:m-1|}$$

$$= \frac{|0:m-1| + \frac{1}{[L,\ldots,L]}}{|1:m-1|} = [a_0, \ldots, a_{m-1}] + \frac{1}{|1:m-1|[L, \ldots, L]},$$

where each sequence $L \oplus \ldots \oplus L$ contains $n-1$ Ls. If m is even, then the plus signs above (of both types!) are replaced by minus signs, and the regular continued fractions (except for $[(a_0, a_1, \ldots, a_{m-1})^n]$ itself) become negative. The theorem follows from taking the limit as $n \to \infty$. □

This theorem is useful because $[L, L, \ldots]$ is easy to compute: it is a positive number x satisfying $x = L + \frac{1}{x}$, and is therefore the positive solution to $x^2 - Lx - 1 = 0$. Similarly, $[L, L, \ldots]_-$ is the positive solution to $x^2 - Lx + 1 = 0$.

Example: Consider $[\overline{3,4,5}]$. All the relevant calculations are: $|0:2| = 68, |1:2| = 21, |1:1| = 4, L = 72, [L, L, \ldots] = 36 + \sqrt{1297}$. Therefore,

$$[\overline{3,4,5}] = \frac{68}{21} + \frac{1}{21(36 + \sqrt{1297})}.$$

Chapter 4

Conclusion and Future Work

In the previous chapter, we were able to use our knowledge of weighted tilings to use in order to derive results about continued fractions. In particular, we learned how to convert between regular and negative continued fractions, the relationships between the convergents, and some results about periodic continued fractions. If the selection of theorems seems arbitrary, it is only because there is so much that could be done. Nearly any problem concerning continued fractions could be attacked using this combinatorial approach. Some examples are given below.

Periodic continued fractions:

This paper has only scratched the surface of periodic continued fractions. For instance, a commonly encountered theorem is that the quadratic surds (i.e. the real numbers of degree 2 over $\mathbb{Q}$) are precisely those numbers whose continued fraction is eventually periodic. Much of the theory for periodic continued fractions can be undoubtedly be extended to negative continued fractions using the approach in this essay.

Two-squares theorems:

Fermat proved the following well-known theorem:

Theorem 35. *Let p be a prime. Then*

$$p = x^2 + y^2$$

for some integers x, y if and only if $p \equiv 1 \mod 4$.

Several other versions of this theorem have been proven since then. The following appear in [Nagell, 1964]:

Theorem 36. *Let $p > 2$ be a prime.*

- *p can be written as $x^2 + y^2$ for some integers x, y if and only if* $p \equiv 1 \mod 4$.
- *p can be written as $x^2 + 3y^2$ if and only if* $p \equiv 1 \mod 6$.
- *p can be written as $x^2 + 2y^2$ if and only if* $p \equiv 1$ or $3 \mod 8$
- *p can be written as $x^2 + 7y^2$ if and only if* $p \equiv 1$ or 9 or $11 \mod 14$.
- *p can be written as $2x^2 + 3y^2$ if and only if* $p \equiv 5$ or $11 \mod 24$

Fermat's original two-squares theorem has a very neat proof, originally due to Smith (1826-1883) involving palindromic continued fractions [Clarke et al., 1999]. The other versions of the theorem may have similar proofs involving palindromic continued fractions with $b_k \neq 1$.

Quadratic Number Fields

Continued fractions often come up in connection with quadratic number fields. For instance, Ireland and Rosen [1990] and Zagier [1981] give some examples of how continued fractions appear when computing class numbers of these fields. Perhaps there is a combinatorial explanation behind this to be discovered.

Bibliography

Arthur T. Benjamin and Jennifer J. Quinn. *Proofs That Really Count: The Art of Combinatorial Proof*, volume 27 of *The Dolciani Mathematical Expositions*. Mathematical Association of America, Washington, DC, 2003. ISBN 0-88385-333-7.

Arthur T. Benjamin and Jeremy A. Rouse. Recounting binomial Fibonacci identities. In *Applications of Fibonacci numbers. Vol. 9*, pages 25–28. Kluwer Acad. Publ., Dordrecht, 2004.

Claude Brezinski. *History of Continued Fractions and Padé Approximants*, volume 12 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1991. ISBN 3-540-15286-5.

F. W. Clarke, W. N. Everitt, L. L. Littlejohn, and S. J. R. Vorster. H. J. S. Smith and the fermat two squares theorem. *Amer. Math. Monthly*, 106(7): 652–665, 1999. ISSN 0002-9890.

Kenneth Ireland and Michael Rosen. *A Classical Introduction to Modern Number Theory*, volume 84 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990. ISBN 0-387-97329-X.

Trygve Nagell. *Introduction to Number Theory*. Second edition. Chelsea Publishing Co., New York, 1964.

Andrew M. Rockett and Peter Szüsz. *Continued fractions*. World Scientific Publishing Co. Inc., River Edge, NJ, 1992. ISBN 981-02-1047-7.

D. B. Zagier. *Zetafunktionen und Quadratische Körper*. Springer-Verlag, Berlin, 1981. ISBN 3-540-10603-0. Eine Einführung in die höhere Zahlentheorie. [An introduction to higher number theory], Hochschultext. [University Text].